Homework #7  Solutions

1)  Diffusive acceleration and its efficiency at the bow shock

a) The observations, i.e. a relatively constant upstream ion flux, suggest a stationary situation. Thus

\[ u_{sw} f(p, x) - \kappa \frac{\partial f(p, x)}{\partial x} = 0 \]

We replace the gradient by \( 1/L \) (with \( L \) the typical length scale of change) and get

\[ u_{sw} - \kappa / L = 0 \]

\[ \kappa = L \cdot u_{sw} \]

In this way we get:

\[
\begin{array}{ccc}
E[\text{keV}] & L[\text{R}_E] & \kappa[\text{m}^2/\text{s}] \\
10 & 3 & 8.5 \cdot 10^{12} \\
20 & 4.4 & 1.2 \cdot 10^{13} \\
30 & 6 & 1.7 \cdot 10^{13} \\
60 & 10 & 2.8 \cdot 10^{13} \\
\end{array}
\]

The typical average diffusion velocity, i.e. the speed with which the particles diffuse into the upstream direction, is equal to \( u_{sw} \). The system sets itself up to balance the convection towards the shock with \( u_{sw} \).

b) We use the average momentum gain

\[
\frac{<\Delta p>}{p} = \frac{2}{3} \frac{u_1 - u_2}{v}
\]

for 1 cycle (after Jones and Ellison)

Let us now assume we would be able to accelerate the ions by Fermi acceleration out of the thermal plasma (which is a drastic and generally not justified simplification). To get a feeling for the power or efficiency of Fermi acceleration this may be allowed here. Then the number of cycles \( n \) to reach the final momentum \( p \) is

\[
n = \frac{p}{<\Delta p>} = \frac{3v}{2(u_1 - u_2)} = \frac{E}{E_{sw}} \frac{3u_{sw}}{2(u_{sw} - u_2)}
\]

assuming for simplicity that we add up \( \Delta p \) for individual particles along \( u_{sw} \). The result looks a little different, if we compute \( <p> \), but the order of magnitude is the same.

With \( r = 4 \) we get

\[
n = \sqrt{\frac{E}{E_{sw}}} \frac{3u_{sw}}{2(u_{sw} - u_2)} = \frac{3r}{2(r-1)} = 2 \sqrt{\frac{E}{E_{sw}}}
\]

In this simplistic manner the particles would need

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 10 \text{ keV} )</th>
<th>( 20 \text{ keV} )</th>
<th>( 30 \text{ keV} ) and ( 60 \text{ keV} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>9</td>
<td>11</td>
<td>16</td>
</tr>
</tbody>
</table>
c) On average a particle has to travel one mean free pathlength \( \lambda \) upstream to be turned around towards the shock. Assuming the mean free path is much shorter in the downstream region, because the diffusion coefficient is smaller, a particle needs

\[
\tau_{\text{acc}} = n \cdot 2 \cdot \lambda_{\text{average}} / v_{\parallel\text{average}}
\]

The mean free path can be derived from the diffusion coefficient:

\[
\lambda = \frac{3k_{\parallel}}{v_{\parallel}} \quad \text{where} \quad v_{\parallel} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{E}{E_{\text{sw}}}} \cdot u_{\text{sw}}
\]

For simplicity we use the average velocity between an initial injection with \( u_{\text{sw}} \) and the final velocity for \( v_{\parallel\text{average}} \). In addition, we recognize that an isotropic distribution has been assumed. Therefore, we set

\[
v_{\parallel\text{average}} = v_{\parallel} / 2
\]

We get

<table>
<thead>
<tr>
<th>E[keV]</th>
<th>( \lambda [R_E] )</th>
<th>( \tau_{\text{acc}} [\text{sec}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.8</td>
<td>500</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>560</td>
</tr>
<tr>
<td>30</td>
<td>3.2</td>
<td>620</td>
</tr>
<tr>
<td>60</td>
<td>3.9</td>
<td>800</td>
</tr>
</tbody>
</table>

It should be noted here that the mean free path is of the order of the e-folding distance of the particles. I.e. the assumption of an isotropic distribution and ideal diffusion is not fully justified! To do better one needs to solve the transport equations more rigorously. This also tells us that the derivation as presented in class has been oversimplified, because a large fraction of the particles will be lost into the upstream direction after each shock crossing cycle. This loss has to be taken into account in a quantitative model.
d) The IMF is convected into the bow shock with $u_{sw}$ IMF $\parallel$ $u_{sw}$

\[ \text{quasi-perpendicular} \quad \text{sub-solar shock} \quad \bullet \quad \text{point} \quad \text{quasi-parallel} \quad \text{Rs shock} \]

For simplicity we assume a spherical shock.

The time of connection with the bow shock for a specific field line is from quasi-perpendicular to sub-solar:

\[ \tau_{\text{subsolar}} = R_s \cdot (1/\sin \phi - 1)/u_{sw} \]

and from quasi-perpendicular to quasi-parallel:

\[ \tau_{\text{quasiparallel}} = R_s / (\sin \phi \cdot u_{sw}) \]

where $\phi$ is the angle between the magnetic field and the solar wind.

For a Parker spiral: $\phi = 45^\circ$, $R_s = 14$ Rs

$\tau_{\text{subsolar}} = 84$ sec

$\tau_{\text{quasiparallel}} = 290$ sec

Even up to the quasi-parallel shock the connection time is not sufficient to accelerate the ions in this model, because we need $\tau_{\text{accel}} \leq \tau_{\text{subsolar}}$ or $\tau_{\text{accel}} \leq \tau_{\text{quasiparallel}}$ to reach these energies. The field needs to be more radial to do the job.

Using this relation, the maximum angle $\phi$ between B and $u_{sw}$ to reach 60 keV is therefore:

Sub-solar: $\sin \phi_{\text{max}} = 1/[(\tau_{\text{accel}} \cdot u_{sw}/R_s + 1]$

$= 0.17 \quad \phi_{\text{max}} \approx 10^\circ$

quasi-parallel:

$\sin \phi_{\text{max}} = R_s / [\tau_{\text{accel}} \cdot u_{sw}]$

$= 0.25 \quad \phi_{\text{max}} \approx 15^\circ$

These estimates show us that Fermi acceleration cannot explain the diffuse upstream particles alone.
2) Diffusive shock (Fermi) acceleration

a) We have derived the following distribution for a particle distribution injected from the upstream region [(3.19) in Jones and Ellison]. We have already neglected the homogeneous term here, because it only contributes, if particles are generated at the shock (for example, pair production), whereas the injection of fresh particles from the solar wind is included in the inhomogeneous term already as $F_1(p')$.

\[
(3.19) \quad F_2(p) = \frac{3r}{r-1} p^{-\delta} \int dp' p'^{\delta-1} F_1(p') \quad \text{where} \quad \delta = \frac{r+2}{r-1}
\]

We inject $F_1(p') = N_o \cdot (p' - p_o)$ with $\frac{p_o}{m} \gg u_1$

\[
F_2(p) = N_o \cdot \frac{3r}{r-1} p^{-\delta} \int dp' p'^{\delta-1} \cdot \delta(p' - p_o) = N_o \cdot \frac{3r}{r-1} p^{-\delta} \cdot \delta(p - p_o)
\]

\[
F_2(p) = N_o \frac{3r}{r-1} p^{-\delta} = N_o \frac{3r}{p_o} \left( \frac{p}{p_o} \right)^{-\frac{r+2}{r-1}} \quad \text{q.e.d.}
\]

b) We know

\[
\frac{dJ}{dEd\Omega} = \frac{v^2}{m} f(v)
\]

from the Lecture Notes

thus

\[
dJ = \frac{v^2}{m} f(v) dEd\Omega
\]

and with $dE = mvdv$

\[
= \frac{v^2}{m} f(v) mvdv d\Omega
\]

Now $f(v) v^2 dv = f(p) p^3 dp$

\[
dJ = p^2 f(p) \frac{mv}{m} dp d\Omega
\]

non-relativistic: $dE = \frac{P}{m} dp$ Thus: $dJ = p^2 f(p) dE d\Omega$ or

\[
\frac{dJ}{dEd\Omega} = p^2 f(p) \quad \text{with} \quad F(p) = 4\pi p^2 f(p)
\]

We insert $F_2(p)$ from 2a)

\[
\frac{dJ}{dEd\Omega} = \frac{F(p)}{4\pi} = \frac{N_o}{4\pi p_o} \left( \frac{3r}{r-1} \right) \left( \frac{p}{p_o} \right)^{-\delta}
\]

With $\frac{p}{p_o} = \sqrt{E/E_o}$ we get
\[
\frac{dJ}{dE d\Omega} = \frac{N_o}{4\pi p_o \frac{3r}{r-1}} \left( \frac{E}{E_o} \right)^{-\frac{\delta}{2}}
\]
or again restricted to non-relativistic cases using \( p_o = 2E_o/v_o \):

\[
\frac{dJ}{dE d\Omega} = \frac{N_o v_o}{8\pi E_o \frac{3r}{r-1}} \left( \frac{E}{E_o} \right)^{-\frac{\delta}{2}}
\]

where \( N_o v_o = j_o \) can be interpreted as the injected flux density into the acceleration process.

d) For a strong shock:

\( M \rightarrow \infty \) \hspace{1cm} \( r \rightarrow 4 \)

i.e.

\[
\delta = \frac{r+2}{r-1} = \frac{6}{3} = 2
\]

The shock spectrum is only determined by the compression ratio. In the limiting case of a strong shock we get a spectrum which scales as

\[
\frac{dJ}{dE} \left( \frac{E}{E_o} \right)^{-1}
\]
3) Energetic particle transport

a) The interplanetary magnetic field lines curve according to the following sketch:

\[ \tan \phi \, dr = \frac{\Omega r}{u_{sw}} \]

Particles travel along field lines, i.e. they travel along the path length.

\[ \int_{0}^{1AU} ds = \int_{0}^{1AU} \sqrt{1 + \tan^2 \phi} \, dr = \int_{0}^{1AU} \frac{\Omega r}{u_{sw}} \, dr \]

We neglect the solar radius for simplicity

\[ R_s << 1AU \]

The angle of the magnetic field at distance r is

\[ \tan \phi = \frac{\Omega r}{u_{sw}} \]

where again \( R_s \) is neglected.

Thus

\[ s = 1AU \int_{0}^{1AU} \sqrt{1 + \left( \frac{\Omega r}{u_{sw}} \right)^2} \, dr = \frac{\Omega \cdot 1AU}{u_{sw}} \int_{0}^{u_{sw}} \sqrt{1 + x^2} \, dx \]

\[ \frac{u_{sw}}{\Omega} = \frac{440 \, \text{km/s}}{2\pi} \cdot 28 \cdot 24 \cdot 3600 \text{ s} \]

where \( x = \frac{\Omega r}{u_{sw}} \) (Solar rotation = 28 days)

\[ = 1.7 \cdot 10^8 \, \text{km} = 1.13 \, \text{AU} \]

\[ s = 1.13 \, \text{AU} \int_{0}^{0.88} \sqrt{1 + x^2} \, dx = 1.13 \, \text{AU} \cdot \frac{1}{2} \left( x \sqrt{x^2 + 1} + \arcsin x \right)_{0}^{0.88} = 1.3 \, \text{AU} \]

The Table below contains the particle speed \( v \), their minimum travel time \( \tau_{\text{transport}} \) along the spiral, while escaping freely, and the two diffusion coefficients for b).
The approximate flare location is at a longitude
\[ \ell = 180/\pi \Omega \times 1 \text{AU} / u_{sw} \]

\[ = 51^\circ \text{ West of the Earth – Sun line} \]
Remember West in the sky according to astronomical convention!

b) The 3-dimensional diffusion coefficient is
\[ \kappa = \frac{\lambda \cdot v}{3} \]

we get \[ = \frac{1.5 \times 10^{11} [\text{m}] \cdot \lambda [\text{AU}] \cdot v [\text{m/sec}]}{3} \]
see the table above.

These diffusion coefficients are much larger than those at the bow shock. Thus the diffusion in the region in front of the bow shock is not caused by the intrinsic interplanetary magnetic field fluctuations in the solar wind. At the shock the diffusing ions themselves create the waves and thus a substantially lower diffusion coefficient.

c) The number of steps \( \lambda \) for a 1-dimensional random walk along the field line is
\[ N = \frac{<x^2>}{\lambda^2} \]
where \( <x^2> \) is the average distance\(^2\) which has been traveled by the particle. In our case we set \( <x^2> = s^2 \) where \( s \) is the length of the field line. Therefore, the effective travel path is
\[ d = N \cdot \lambda = s^2 / \lambda \]
and the average flight time:
\[ \tau_{\text{average}} = \frac{d}{v} = \frac{s^2}{(\lambda v)} \]

<table>
<thead>
<tr>
<th>E[keV/Nuc]</th>
<th>( \tau_{\text{sec}} )</th>
<th>( \tau_{[h]} )</th>
<th>days</th>
<th>( \tau_{[h]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5.8 \times 10^6</td>
<td>160</td>
<td>6.7</td>
<td>16</td>
</tr>
<tr>
<td>1</td>
<td>1.8 \times 10^6</td>
<td>50</td>
<td>2.1</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>5.8 \times 10^5</td>
<td>16</td>
<td>0.6</td>
<td>1.6</td>
</tr>
</tbody>
</table>

The convection with the solar wind takes about 4 days for 440 km/sec. Except for 100 keV/N ions and \( \lambda = 0.1 \text{ AU} \) the energetic ions are faster with diffusive transport.